REVIEW OF ALGEBRA

Here we review the basic rules and procedures of algebra that you need to know in order to be successful in calculus.

ARITHMETIC OPERATIONS

The real numbers have the following properties:

$a + b = b + a$	$ab = ba$	(Commutative Law)
$(a + b) + c = a + (b + c)$	$(ab)c = a(bc)$	(Associative Law)
$a(b + c) = ab + ac$		(Distributive law)

In particular, putting $a = -1$ in the Distributive Law, we get

$$-(b + c) = (-1)(b + c) = (-1)b + (-1)c$$

and so

$$\boxed{-(b + c) = -b - c}$$

EXAMPLE 1

(a) $(3xy)(-4x) = 3(-4)x^2y = -12x^2y$

(b) $2t(7x + 2tx - 11) = 14tx + 4t^2x - 22t$

(c) $4 - 3(x - 2) = 4 - 3x + 6 = 10 - 3x$ ■

If we use the Distributive Law three times, we get

$$(a + b)(c + d) = (a + b)c + (a + b)d = ac + bc + ad + bd$$

This says that we multiply two factors by multiplying each term in one factor by each term in the other factor and adding the products. Schematically, we have

$$(a + b)(c + d)$$

In the case where $c = a$ and $d = b$, we have

$$(a + b)^2 = a^2 + ba + ab + b^2$$

or

1 $$\boxed{(a + b)^2 = a^2 + 2ab + b^2}$$

Similarly, we obtain

2 $$\boxed{(a - b)^2 = a^2 - 2ab + b^2}$$

EXAMPLE 2

(a) $(2x + 1)(3x - 5) = 6x^2 + 3x - 10x - 5 = 6x^2 - 7x - 5$

(b) $(x + 6)^2 = x^2 + 12x + 36$

(c) $3(x - 1)(4x + 3) - 2(x + 6) = 3(4x^2 - x - 3) - 2x - 12$

$\qquad = 12x^2 - 3x - 9 - 2x - 12 = 12x^2 - 5x - 21$ ■

FRACTIONS

To add two fractions with the same denominator, we use the Distributive Law:

$$\frac{a}{b} + \frac{c}{b} = \frac{1}{b} \times a + \frac{1}{b} \times c = \frac{1}{b}(a + c) = \frac{a + c}{b}$$

Thus, it is true that

$$\boxed{\frac{a + c}{b} = \frac{a}{b} + \frac{c}{b}}$$

But remember to avoid the following common error:

$$\frac{a}{b + c} \cancel{=} \frac{a}{b} + \frac{a}{c}$$

(For instance, take $a = b = c = 1$ to see the error.)

To add two fractions with different denominators, we use a common denominator:

$$\boxed{\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}}$$

We multiply such fractions as follows:

$$\boxed{\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}}$$

In particular, it is true that

$$\boxed{\frac{-a}{b} = -\frac{a}{b} = \frac{a}{-b}}$$

To divide two fractions, we invert and multiply:

$$\boxed{\frac{\dfrac{a}{b}}{\dfrac{c}{d}} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}}$$

EXAMPLE 3

(a) $\dfrac{x + 3}{x} = \dfrac{x}{x} + \dfrac{3}{x} = 1 + \dfrac{3}{x}$

(b) $\dfrac{3}{x - 1} + \dfrac{x}{x + 2} = \dfrac{3(x + 2) + x(x - 1)}{(x - 1)(x + 2)} = \dfrac{3x + 6 + x^2 - x}{x^2 + x - 2} = \dfrac{x^2 + 2x + 6}{x^2 + x - 2}$

(c) $\dfrac{s^2t}{u} \cdot \dfrac{ut}{-2} = \dfrac{s^2t^2u}{-2u} = -\dfrac{s^2t^2}{2}$

(d) $$\frac{\dfrac{x}{y}+1}{1-\dfrac{y}{x}} = \frac{\dfrac{x+y}{y}}{\dfrac{x-y}{x}} = \frac{x+y}{y} \times \frac{x}{x-y} = \frac{x(x+y)}{y(x-y)} = \frac{x^2+xy}{xy-y^2}$$ ■

FACTORING

We have used the Distributive Law to expand certain algebraic expressions. We sometimes need to reverse this process (again using the Distributive Law) by factoring an expression as a product of simpler ones. The easiest situation occurs when the expression has a common factor as follows:

$$\xrightarrow{\text{Expanding}}$$
$$3x(x-2) = 3x^2 - 6x$$
$$\xleftarrow{\text{Factoring}}$$

To factor a quadratic of the form $x^2 + bx + c$ we note that

$$(x+r)(x+s) = x^2 + (r+s)x + rs$$

so we need to choose numbers r and s so that $r + s = b$ and $rs = c$.

EXAMPLE 4 Factor $x^2 + 5x - 24$.

SOLUTION The two integers that add to give 5 and multiply to give -24 are -3 and 8. Therefore

$$x^2 + 5x - 24 = (x-3)(x+8)$$ ■

EXAMPLE 5 Factor $2x^2 - 7x - 4$.

SOLUTION Even though the coefficient of x^2 is not 1, we can still look for factors of the form $2x + r$ and $x + s$, where $rs = -4$. Experimentation reveals that

$$2x^2 - 7x - 4 = (2x+1)(x-4)$$ ■

Some special quadratics can be factored by using Equations 1 or 2 (from right to left) or by using the formula for a difference of squares:

3 $$\boxed{a^2 - b^2 = (a-b)(a+b)}$$

The analogous formula for a difference of cubes is

4 $$\boxed{a^3 - b^3 = (a-b)(a^2 + ab + b^2)}$$

which you can verify by expanding the right side. For a sum of cubes we have

5 $$\boxed{a^3 + b^3 = (a+b)(a^2 - ab + b^2)}$$

EXAMPLE 6

(a) $x^2 - 6x + 9 = (x-3)^2$ (Equation 2; $a = x, b = 3$)

(b) $4x^2 - 25 = (2x-5)(2x+5)$ (Equation 3; $a = 2x, b = 5$)

(c) $x^3 + 8 = (x+2)(x^2 - 2x + 4)$ (Equation 5; $a = x, b = 2$) ■

EXAMPLE 7 Simplify $\dfrac{x^2 - 16}{x^2 - 2x - 8}$.

SOLUTION Factoring numerator and denominator, we have

$$\frac{x^2 - 16}{x^2 - 2x - 8} = \frac{(x - 4)(x + 4)}{(x - 4)(x + 2)} = \frac{x + 4}{x + 2}$$ ■

To factor polynomials of degree 3 or more, we sometimes use the following fact.

> **6 The Factor Theorem** If P is a polynomial and $P(b) = 0$, then $x - b$ is a factor of $P(x)$.

EXAMPLE 8 Factor $x^3 - 3x^2 - 10x + 24$.

SOLUTION Let $P(x) = x^3 - 3x^2 - 10x + 24$. If $P(b) = 0$, where b is an integer, then b is a factor of 24. Thus, the possibilities for b are $\pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12$, and ±24. We find that $P(1) = 12$, $P(-1) = 30$, $P(2) = 0$. By the Factor Theorem, $x - 2$ is a factor. Instead of substituting further, we use long division as follows:

$$\begin{array}{r|l}
 & x^2 - x - 12 \\
\hline
x - 2 & x^3 - 3x^2 - 10x + 24 \\
 & x^3 - 2x^2 \\
\hline
 & \quad -x^2 - 10x \\
 & \quad -x^2 + 2x \\
\hline
 & \qquad -12x + 24 \\
 & \qquad -12x + 24 \\
\hline
\end{array}$$

Therefore
$$\begin{aligned} x^3 - 3x^2 - 10x + 24 &= (x - 2)(x^2 - x - 12) \\ &= (x - 2)(x + 3)(x - 4) \end{aligned}$$ ■

COMPLETING THE SQUARE

Completing the square is a useful technique for graphing parabolas or integrating rational functions. Completing the square means rewriting a quadratic $ax^2 + bx + c$ in the form $a(x + p)^2 + q$ and can be accomplished by:

1. Factoring the number a from the terms involving x.
2. Adding and subtracting the square of half the coefficient of x.

In general, we have

$$\begin{aligned} ax^2 + bx + c &= a\left[x^2 + \frac{b}{a}x\right] + c \\ &= a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right] + c \\ &= a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right) \end{aligned}$$

EXAMPLE 9 Rewrite $x^2 + x + 1$ by completing the square.

SOLUTION The square of half the coefficient of x is $\frac{1}{4}$. Thus

$$x^2 + x + 1 = x^2 + x + \tfrac{1}{4} - \tfrac{1}{4} + 1 = \left(x + \tfrac{1}{2}\right)^2 + \tfrac{3}{4}$$ ■

EXAMPLE 10

$$2x^2 - 12x + 11 = 2[x^2 - 6x] + 11 = 2[x^2 - 6x + 9 - 9] + 11$$
$$= 2[(x - 3)^2 - 9] + 11 = 2(x - 3)^2 - 7$$ ■

QUADRATIC FORMULA

By completing the square as above we can obtain the following formula for the roots of a quadratic equation.

7 The Quadratic Formula The roots of the quadratic equation $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

EXAMPLE 11 Solve the equation $5x^2 + 3x - 3 = 0$.

SOLUTION With $a = 5$, $b = 3$, $c = -3$, the quadratic formula gives the solutions

$$x = \frac{-3 \pm \sqrt{3^2 - 4(5)(-3)}}{2(5)} = \frac{-3 \pm \sqrt{69}}{10}$$ ■

The quantity $b^2 - 4ac$ that appears in the quadratic formula is called the **discriminant**. There are three possibilities:

1. If $b^2 - 4ac > 0$, the equation has two real roots.
2. If $b^2 - 4ac = 0$, the roots are equal.
3. If $b^2 - 4ac < 0$, the equation has no real root. (The roots are complex.)

These three cases correspond to the fact that the number of times the parabola $y = ax^2 + bx + c$ crosses the x-axis is 2, 1, or 0 (see Figure 1). In case (3) the quadratic $ax^2 + bx + c$ can't be factored and is called **irreducible.**

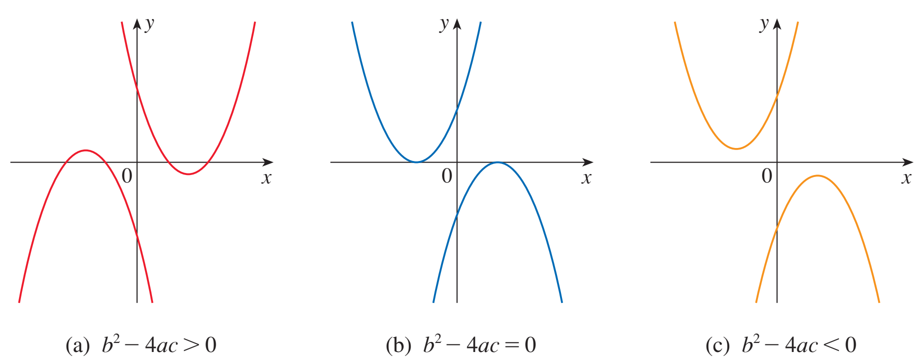


FIGURE 1
Possible graphs of $y = ax^2 + bx + c$

EXAMPLE 12 The quadratic $x^2 + x + 2$ is irreducible because its discriminant is negative:

$$b^2 - 4ac = 1^2 - 4(1)(2) = -7 < 0$$

Therefore, it is impossible to factor $x^2 + x + 2$. ■

THE BINOMIAL THEOREM

Recall the binomial expression from Equation 1:

$$(a + b)^2 = a^2 + 2ab + b^2$$

If we multiply both sides by $(a + b)$ and simplify, we get the binomial expansion

8 $$\boxed{(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3}$$

Repeating this procedure, we get

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

In general, we have the following formula.

9 The Binomial Theorem If k is a positive integer, then

$$\begin{aligned}(a + b)^k = a^k + ka^{k-1}b &+ \frac{k(k-1)}{1 \cdot 2} a^{k-2}b^2 \\ &+ \frac{k(k-1)(k-2)}{1 \cdot 2 \cdot 3} a^{k-3}b^3 \\ &+ \cdots + \frac{k(k-1)\cdots(k-n+1)}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n} a^{k-n}b^n \\ &+ \cdots + kab^{k-1} + b^k\end{aligned}$$

EXAMPLE 13 Expand $(x - 2)^5$.

SOLUTION Using the Binomial Theorem with $a = x$, $b = -2$, $k = 5$, we have

$$\begin{aligned}(x - 2)^5 &= x^5 + 5x^4(-2) + \frac{5 \cdot 4}{1 \cdot 2} x^3(-2)^2 + \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} x^2(-2)^3 + 5x(-2)^4 + (-2)^5 \\ &= x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32\end{aligned}$$

■

RADICALS

The most commonly occurring radicals are square roots. The symbol $\sqrt{}$ means "the positive square root of." Thus

$$x = \sqrt{a} \qquad \text{means} \qquad x^2 = a \quad \text{and} \quad x \geqslant 0$$

Since $a = x^2 \geqslant 0$, the symbol $\sqrt{a}$ makes sense only when $a \geqslant 0$. Here are two rules for working with square roots:

10 $$\boxed{\sqrt{ab} = \sqrt{a}\,\sqrt{b} \qquad \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}}$$

However, there is no similar rule for the square root of a sum. In fact, you should remember to avoid the following common error:

$$\sqrt{a + b} = \sqrt{a} + \sqrt{b}$$

(For instance, take $a = 9$ and $b = 16$ to see the error.)

EXAMPLE 14

(a) $\dfrac{\sqrt{18}}{\sqrt{2}} = \sqrt{\dfrac{18}{2}} = \sqrt{9} = 3$ (b) $\sqrt{x^2 y} = \sqrt{x^2}\sqrt{y} = |x|\sqrt{y}$

Notice that $\sqrt{x^2} = |x|$ because $\sqrt{\ }$ indicates the positive square root. (See **Absolute Value**.) ■

In general, if n is a positive integer,

$$x = \sqrt[n]{a} \qquad \text{means} \qquad x^n = a$$

If n is even, then $a \geqslant 0$ and $x \geqslant 0$.

Thus $\sqrt[3]{-8} = -2$ because $(-2)^3 = -8$, but $\sqrt[4]{-8}$ and $\sqrt[6]{-8}$ are not defined. The following rules are valid:

$$\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b} \qquad\qquad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

EXAMPLE 15 $\sqrt[3]{x^4} = \sqrt[3]{x^3 x} = \sqrt[3]{x^3}\,\sqrt[3]{x} = x\sqrt[3]{x}$ ■

To **rationalize** a numerator or denominator that contains an expression such as $\sqrt{a} - \sqrt{b}$, we multiply both the numerator and the denominator by the conjugate radical $\sqrt{a} + \sqrt{b}$. Then we can take advantage of the formula for a difference of squares:

$$\left(\sqrt{a} - \sqrt{b}\right)\left(\sqrt{a} + \sqrt{b}\right) = \left(\sqrt{a}\right)^2 - \left(\sqrt{b}\right)^2 = a - b$$

EXAMPLE 16 Rationalize the numerator in the expression $\dfrac{\sqrt{x+4} - 2}{x}$.

SOLUTION We multiply the numerator and the denominator by the conjugate radical $\sqrt{x+4} + 2$:

$$\frac{\sqrt{x+4} - 2}{x} = \left(\frac{\sqrt{x+4} - 2}{x}\right)\left(\frac{\sqrt{x+4} + 2}{\sqrt{x+4} + 2}\right) = \frac{(x+4) - 4}{x\left(\sqrt{x+4} + 2\right)}$$

$$= \frac{x}{x\left(\sqrt{x+4} + 2\right)} = \frac{1}{\sqrt{x+4} + 2}$$ ■

EXPONENTS

Let a be any positive number and let n be a positive integer. Then, by definition,

1. $a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}$

2. $a^0 = 1$

3. $a^{-n} = \dfrac{1}{a^n}$

4. $a^{1/n} = \sqrt[n]{a}$

$a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m \qquad m$ is any integer

11 Laws of Exponents Let a and b be positive numbers and let r and s be any rational numbers (that is, ratios of integers). Then

1. $a^r \times a^s = a^{r+s}$ **2.** $\dfrac{a^r}{a^s} = a^{r-s}$ **3.** $(a^r)^s = a^{rs}$

4. $(ab)^r = a^r b^r$ **5.** $\left(\dfrac{a}{b}\right)^r = \dfrac{a^r}{b^r} \quad b \neq 0$

In words, these five laws can be stated as follows:

1. To multiply two powers of the same number, we add the exponents.
2. To divide two powers of the same number, we subtract the exponents.
3. To raise a power to a new power, we multiply the exponents.
4. To raise a product to a power, we raise each factor to the power.
5. To raise a quotient to a power, we raise both numerator and denominator to the power.

EXAMPLE 17

(a) $2^8 \times 8^2 = 2^8 \times (2^3)^2 = 2^8 \times 2^6 = 2^{14}$

(b) $$\frac{x^{-2} - y^{-2}}{x^{-1} + y^{-1}} = \frac{\dfrac{1}{x^2} - \dfrac{1}{y^2}}{\dfrac{1}{x} + \dfrac{1}{y}} = \frac{\dfrac{y^2 - x^2}{x^2y^2}}{\dfrac{y + x}{xy}} = \frac{y^2 - x^2}{x^2y^2} \cdot \frac{xy}{y + x}$$
$$= \frac{(y - x)(y + x)}{xy(y + x)} = \frac{y - x}{xy}$$

(c) $4^{3/2} = \sqrt{4^3} = \sqrt{64} = 8$ Alternative solution: $4^{3/2} = \left(\sqrt{4}\right)^3 = 2^3 = 8$

(d) $\dfrac{1}{\sqrt[3]{x^4}} = \dfrac{1}{x^{4/3}} = x^{-4/3}$

(e) $\left(\dfrac{x}{y}\right)^3 \left(\dfrac{y^2x}{z}\right)^4 = \dfrac{x^3}{y^3} \cdot \dfrac{y^8x^4}{z^4} = x^7y^5z^{-4}$ ■

INEQUALITIES

When working with inequalities, note the following rules.

Rules for Inequalities

1. If $a < b$, then $a + c < b + c$.
2. If $a < b$ and $c < d$, then $a + c < b + d$.
3. If $a < b$ and $c > 0$, then $ac < bc$.
4. If $a < b$ and $c < 0$, then $ac > bc$.
5. If $0 < a < b$, then $1/a > 1/b$.

Rule 1 says that we can add any number to both sides of an inequality, and Rule 2 says that two inequalities can be added. However, we have to be careful with multiplication. Rule 3 says that we can multiply both sides of an inequality by a *positive* number, but Rule 4 says that *if we multiply both sides of an inequality by a negative number, then we reverse the direction of the inequality.* For example, if we take the inequality

$3 < 5$ and multiply by 2, we get $6 < 10$, but if we multiply by -2, we get $-6 > -10$. Finally, Rule 5 says that if we take reciprocals, then we reverse the direction of an inequality (provided the numbers are positive).

EXAMPLE 18 Solve the inequality $1 + x < 7x + 5$.

SOLUTION The given inequality is satisfied by some values of x but not by others. To *solve* an inequality means to determine the set of numbers x for which the inequality is true. This is called the *solution set.*

First we subtract 1 from each side of the inequality (using Rule 1 with $c = -1$):

$$x < 7x + 4$$

Then we subtract $7x$ from both sides (Rule 1 with $c = -7x$):

$$-6x < 4$$

Now we divide both sides by -6 $\left(\text{Rule 4 with } c = -\frac{1}{6}\right)$:

$$x > -\tfrac{4}{6} = -\tfrac{2}{3}$$

These steps can all be reversed, so the solution set consists of all numbers greater than $-\frac{2}{3}$. In other words, the solution of the inequality is the interval $\left(-\frac{2}{3}, \infty\right)$. ■

EXAMPLE 19 Solve the inequality $x^2 - 5x + 6 \leqslant 0$.

SOLUTION First we factor the left side:

$$(x - 2)(x - 3) \leqslant 0$$

We know that the corresponding equation $(x - 2)(x - 3) = 0$ has the solutions 2 and 3. The numbers 2 and 3 divide the real line into three intervals:

$$(-\infty, 2) \qquad (2, 3) \qquad (3, \infty)$$

On each of these intervals we determine the signs of the factors. For instance,

$$x \in (-\infty, 2) \quad \Rightarrow \quad x < 2 \quad \Rightarrow \quad x - 2 < 0$$

Then we record these signs in the following chart:

Interval	$x - 2$	$x - 3$	$(x - 2)(x - 3)$
$x < 2$	$-$	$-$	$+$
$2 < x < 3$	$+$	$-$	$-$
$x > 3$	$+$	$+$	$+$

■■ A visual method for solving Example 19 is to use a graphing device to graph the parabola $y = x^2 - 5x + 6$ (as in Figure 2) and observe that the curve lies on or below the x-axis when $2 \leqslant x \leqslant 3$.

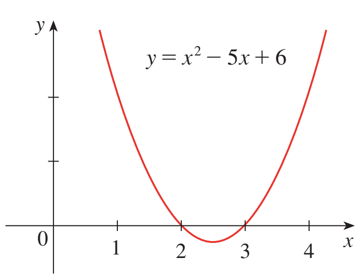


FIGURE 2

Another method for obtaining the information in the chart is to use *test values.* For instance, if we use the test value $x = 1$ for the interval $(-\infty, 2)$, then substitution in $x^2 - 5x + 6$ gives

$$1^2 - 5(1) + 6 = 2$$

The polynomial $x^2 - 5x + 6$ doesn't change sign inside any of the three intervals, so we conclude that it is positive on $(-\infty, 2)$.

Then we read from the chart that $(x - 2)(x - 3)$ is negative when $2 < x < 3$. Thus, the solution of the inequality $(x - 2)(x - 3) \leqslant 0$ is

$$\{x \mid 2 \leqslant x \leqslant 3\} = [2, 3]$$

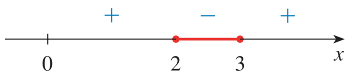


FIGURE 3

Notice that we have included the endpoints 2 and 3 because we are looking for values of x such that the product is either negative or zero. The solution is illustrated in Figure 3. ■

EXAMPLE 20 Solve $x^3 + 3x^2 > 4x$.

SOLUTION First we take all nonzero terms to one side of the inequality sign and factor the resulting expression:

$$x^3 + 3x^2 - 4x > 0 \qquad \text{or} \qquad x(x-1)(x+4) > 0$$

As in Example 19 we solve the corresponding equation $x(x-1)(x+4) = 0$ and use the solutions $x = -4$, $x = 0$, and $x = 1$ to divide the real line into four intervals $(-\infty, -4)$, $(-4, 0)$, $(0, 1)$, and $(1, \infty)$. On each interval the product keeps a constant sign as shown in the following chart.

Interval	x	$x - 1$	$x + 4$	$x(x-1)(x+4)$
$x < -4$	−	−	−	−
$-4 < x < 0$	−	−	+	+
$0 < x < 1$	+	−	+	−
$x > 1$	+	+	+	+

Then we read from the chart that the solution set is

$$\{x \mid -4 < x < 0 \text{ or } x > 1\} = (-4, 0) \cup (1, \infty)$$

FIGURE 4

The solution is illustrated in Figure 4. ■

ABSOLUTE VALUE

The **absolute value** of a number a, denoted by $|a|$, is the distance from a to 0 on the real number line. Distances are always positive or 0, so we have

$$|a| \geqslant 0 \qquad \text{for every number } a$$

For example,

$$|3| = 3 \qquad |-3| = 3 \qquad |0| = 0$$

$$|\sqrt{2} - 1| = \sqrt{2} - 1 \qquad |3 - \pi| = \pi - 3$$

In general, we have

■■ Remember that if a is negative, then $-a$ is positive.

12

$$\boxed{\begin{aligned} |a| &= a && \text{if } a \geqslant 0 \\ |a| &= -a && \text{if } a < 0 \end{aligned}}$$

EXAMPLE 21 Express $|3x - 2|$ without using the absolute-value symbol.

SOLUTION

$$|3x - 2| = \begin{cases} 3x - 2 & \text{if } 3x - 2 \geqslant 0 \\ -(3x - 2) & \text{if } 3x - 2 < 0 \end{cases}$$

$$= \begin{cases} 3x - 2 & \text{if } x \geqslant \frac{2}{3} \\ 2 - 3x & \text{if } x < \frac{2}{3} \end{cases}$$ ■

Recall that the symbol $\sqrt{\ }$ means "the positive square root of." Thus, $\sqrt{r} = s$ means $s^2 = r$ and $s \geqslant 0$. Therefore, the equation $\sqrt{a^2} = a$ is not always true. It is true only when $a \geqslant 0$. If $a < 0$, then $-a > 0$, so we have $\sqrt{a^2} = -a$. In view of (12), we then have the equation

13
$$\boxed{\sqrt{a^2} = |a|}$$

which is true for all values of a.

Hints for the proofs of the following properties are given in the exercises.

Properties of Absolute Values Suppose a and b are any real numbers and n is an integer. Then

1. $|ab| = |a||b|$ **2.** $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|} \quad (b \neq 0)$ **3.** $|a^n| = |a|^n$

For solving equations or inequalities involving absolute values, it's often very helpful to use the following statements.

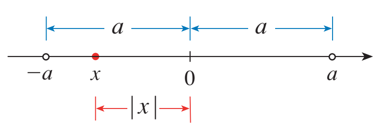


FIGURE 5

Suppose $a > 0$. Then

4. $|x| = a$ if and only if $x = \pm a$

5. $|x| < a$ if and only if $-a < x < a$

6. $|x| > a$ if and only if $x > a$ or $x < -a$

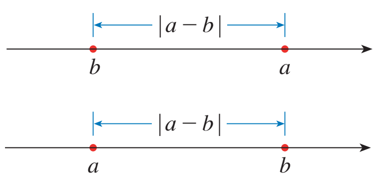


FIGURE 6

Length of a line segment $= |a - b|$

For instance, the inequality $|x| < a$ says that the distance from x to the origin is less than a, and you can see from Figure 5 that this is true if and only if x lies between $-a$ and a.

If a and b are any real numbers, then the distance between a and b is the absolute value of the difference, namely, $|a - b|$, which is also equal to $|b - a|$. (See Figure 6.)

EXAMPLE 22 Solve

$$|2x - 5| = 3$$

SOLUTION By Property 4 of absolute values, $|2x - 5| = 3$ is equivalent to

$$2x - 5 = 3 \qquad \text{or} \qquad 2x - 5 = -3$$

So $2x = 8$ or $2x = 2$. Thus, $x = 4$ or $x = 1$. ■

EXAMPLE 23 Solve $|x - 5| < 2$.

SOLUTION 1 By Property 5 of absolute values, $|x - 5| < 2$ is equivalent to

$$-2 < x - 5 < 2$$

Therefore, adding 5 to each side, we have

$$3 < x < 7$$

and the solution set is the open interval $(3, 7)$.

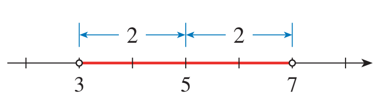


FIGURE 7

SOLUTION 2 Geometrically, the solution set consists of all numbers x whose distance from 5 is less than 2. From Figure 7 we see that this is the interval $(3, 7)$. ■

EXAMPLE 24 Solve $|3x + 2| \geqslant 4$.

SOLUTION By Properties 4 and 6 of absolute values, $|3x + 2| \geqslant 4$ is equivalent to

$$3x + 2 \geqslant 4 \qquad \text{or} \qquad 3x + 2 \leqslant -4$$

In the first case, $3x \geqslant 2$, which gives $x \geqslant \frac{2}{3}$. In the second case, $3x \leqslant -6$, which gives $x \leqslant -2$. So the solution set is

$$\left\{x \mid x \leqslant -2 \text{ or } x \geqslant \tfrac{2}{3}\right\} = (-\infty, -2] \cup \left[\tfrac{2}{3}, \infty\right)$$ ■

EXERCISES

A Click here for answers. S Click here for solutions.

1–16 Expand and simplify.

1. $(-6ab)(0.5ac)$
2. $-(2x^2y)(-xy^4)$
3. $2x(x - 5)$
4. $(4 - 3x)x$
5. $-2(4 - 3a)$
6. $8 - (4 + x)$
7. $4(x^2 - x + 2) - 5(x^2 - 2x + 1)$
8. $5(3t - 4) - (t^2 + 2) - 2t(t - 3)$
9. $(4x - 1)(3x + 7)$
10. $x(x - 1)(x + 2)$
11. $(2x - 1)^2$
12. $(2 + 3x)^2$
13. $y^4(6 - y)(5 + y)$
14. $(t - 5)^2 - 2(t + 3)(8t - 1)$
15. $(1 + 2x)(x^2 - 3x + 1)$
16. $(1 + x - x^2)^2$

17–28 Perform the indicated operations and simplify.

17. $\dfrac{2 + 8x}{2}$
18. $\dfrac{9b - 6}{3b}$
19. $\dfrac{1}{x + 5} + \dfrac{2}{x - 3}$
20. $\dfrac{1}{x + 1} + \dfrac{1}{x - 1}$
21. $u + 1 + \dfrac{u}{u + 1}$
22. $\dfrac{2}{a^2} - \dfrac{3}{ab} + \dfrac{4}{b^2}$
23. $\dfrac{x/y}{z}$
24. $\dfrac{x}{y/z}$
25. $\left(\dfrac{-2r}{s}\right)\left(\dfrac{s^2}{-6t}\right)$
26. $\dfrac{a}{bc} \div \dfrac{b}{ac}$
27. $\dfrac{1 + \dfrac{1}{c - 1}}{1 - \dfrac{1}{c - 1}}$
28. $1 + \dfrac{1}{1 + \dfrac{1}{1 + x}}$

29–48 Factor the expression.

29. $2x + 12x^3$
30. $5ab - 8abc$
31. $x^2 + 7x + 6$
32. $x^2 - x - 6$
33. $x^2 - 2x - 8$
34. $2x^2 + 7x - 4$
35. $9x^2 - 36$
36. $8x^2 + 10x + 3$
37. $6x^2 - 5x - 6$
38. $x^2 + 10x + 25$
39. $t^3 + 1$
40. $4t^2 - 9s^2$
41. $4t^2 - 12t + 9$
42. $x^3 - 27$
43. $x^3 + 2x^2 + x$
44. $x^3 - 4x^2 + 5x - 2$
45. $x^3 + 3x^2 - x - 3$
46. $x^3 - 2x^2 - 23x + 60$
47. $x^3 + 5x^2 - 2x - 24$
48. $x^3 - 3x^2 - 4x + 12$

49–54 Simplify the expression.

49. $\dfrac{x^2 + x - 2}{x^2 - 3x + 2}$
50. $\dfrac{2x^2 - 3x - 2}{x^2 - 4}$
51. $\dfrac{x^2 - 1}{x^2 - 9x + 8}$
52. $\dfrac{x^3 + 5x^2 + 6x}{x^2 - x - 12}$
53. $\dfrac{1}{x + 3} + \dfrac{1}{x^2 - 9}$
54. $\dfrac{x}{x^2 + x - 2} - \dfrac{2}{x^2 - 5x + 4}$

55–60 Complete the square.

55. $x^2 + 2x + 5$
56. $x^2 - 16x + 80$
57. $x^2 - 5x + 10$
58. $x^2 + 3x + 1$
59. $4x^2 + 4x - 2$
60. $3x^2 - 24x + 50$

61–68 Solve the equation.

61. $x^2 + 9x - 10 = 0$
62. $x^2 - 2x - 8 = 0$
63. $x^2 + 9x - 1 = 0$
64. $x^2 - 2x - 7 = 0$
65. $3x^2 + 5x + 1 = 0$
66. $2x^2 + 7x + 2 = 0$
67. $x^3 - 2x + 1 = 0$
68. $x^3 + 3x^2 + x - 1 = 0$

69–72 Which of the quadratics are irreducible?

69. $2x^2 + 3x + 4$
70. $2x^2 + 9x + 4$
71. $3x^2 + x - 6$
72. $x^2 + 3x + 6$

73–76 Use the Binomial Theorem to expand the expression.

73. $(a + b)^6$
74. $(a + b)^7$
75. $(x^2 - 1)^4$
76. $(3 + x^2)^5$

77–82 ■ Simplify the radicals.

77. $\sqrt{32}\,\sqrt{2}$ **78.** $\dfrac{\sqrt[3]{-2}}{\sqrt[3]{54}}$ **79.** $\dfrac{\sqrt[4]{32x^4}}{\sqrt[4]{2}}$

80. $\sqrt{xy}\,\sqrt{x^3y}$ **81.** $\sqrt{16a^4b^3}$ **82.** $\dfrac{\sqrt[5]{96a^6}}{\sqrt[5]{3a}}$

83–100 Use the Laws of Exponents to rewrite and simplify the expression.

83. $3^{10} \times 9^8$ **84.** $2^{16} \times 4^{10} \times 16^6$

85. $\dfrac{x^9(2x)^4}{x^3}$ **86.** $\dfrac{a^n \times a^{2n+1}}{a^{n-2}}$

87. $\dfrac{a^{-3}b^4}{a^{-5}b^5}$ **88.** $\dfrac{x^{-1} + y^{-1}}{(x + y)^{-1}}$

89. $3^{-1/2}$ **90.** $96^{1/5}$

91. $125^{2/3}$ **92.** $64^{-4/3}$

93. $(2x^2y^4)^{3/2}$ **94.** $(x^{-5}y^3z^{10})^{-3/5}$

95. $\sqrt[5]{y^6}$ **96.** $(\sqrt[4]{a})^3$

97. $\dfrac{1}{(\sqrt{t})^5}$ **98.** $\dfrac{\sqrt[8]{x^5}}{\sqrt[4]{x^3}}$

99. $\sqrt[4]{\dfrac{t^{1/2}\sqrt{st}}{s^{2/3}}}$ **100.** $\sqrt[4]{r^{2n+1}} \times \sqrt[4]{r^{-1}}$

101–108 Rationalize the expression.

101. $\dfrac{\sqrt{x} - 3}{x - 9}$ **102.** $\dfrac{(1/\sqrt{x}) - 1}{x - 1}$

103. $\dfrac{x\sqrt{x} - 8}{x - 4}$ **104.** $\dfrac{\sqrt{2 + h} + \sqrt{2 - h}}{h}$

105. $\dfrac{2}{3 - \sqrt{5}}$ **106.** $\dfrac{1}{\sqrt{x} - \sqrt{y}}$

107. $\sqrt{x^2 + 3x + 4} - x$ **108.** $\sqrt{x^2 + x} - \sqrt{x^2 - x}$

109–116 State whether or not the equation is true for all values of the variable.

109. $\sqrt{x^2} = x$ **110.** $\sqrt{x^2 + 4} = |x| + 2$

111. $\dfrac{16 + a}{16} = 1 + \dfrac{a}{16}$ **112.** $\dfrac{1}{x^{-1} + y^{-1}} = x + y$

113. $\dfrac{x}{x + y} = \dfrac{1}{1 + y}$ **114.** $\dfrac{2}{4 + x} = \dfrac{1}{2} + \dfrac{2}{x}$

115. $(x^3)^4 = x^7$

116. $6 - 4(x + a) = 6 - 4x - 4a$

117–126 Rewrite the expression without using the absolute value symbol.

117. $|5 - 23|$ **118.** $|\pi - 2|$

119. $|\sqrt{5} - 5|$ **120.** $||-2| - |-3||$

121. $|x - 2|$ if $x < 2$ **122.** $|x - 2|$ if $x > 2$

123. $|x + 1|$ **124.** $|2x - 1|$

125. $|x^2 + 1|$ **126.** $|1 - 2x^2|$

127–142 Solve the inequality in terms of intervals and illustrate the solution set on the real number line.

127. $2x + 7 > 3$ **128.** $4 - 3x \geqslant 6$

129. $1 - x \leqslant 2$ **130.** $1 + 5x > 5 - 3x$

131. $0 \leqslant 1 - x < 1$ **132.** $1 < 3x + 4 \leqslant 16$

133. $(x - 1)(x - 2) > 0$ **134.** $x^2 < 2x + 8$

135. $x^2 < 3$ **136.** $x^2 \geqslant 5$

137. $x^3 - x^2 \leqslant 0$

138 $(x + 1)(x - 2)(x + 3) \geqslant 0$

139. $x^3 > x$ **140.** $x^3 + 3x < 4x^2$

141. $\dfrac{1}{x} < 4$ **142.** $-3 < \dfrac{1}{x} \leqslant 1$

143. The relationship between the Celsius and Fahrenheit temperature scales is given by $C = \frac{5}{9}(F - 32)$, where C is the temperature in degrees Celsius and F is the temperature in degrees Fahrenheit. What interval on the Celsius scale corresponds to the temperature range $50 \leqslant F \leqslant 95$?

144. Use the relationship between C and F given in Exercise 143 to find the interval on the Fahrenheit scale corresponding to the temperature range $20 \leqslant C \leqslant 30$.

145. As dry air moves upward, it expands and in so doing cools at a rate of about 1°C for each 100-m rise, up to about 12 km.
(a) If the ground temperature is 20°C, write a formula for the temperature at height h.
(b) What range of temperature can be expected if a plane takes off and reaches a maximum height of 5 km?

146. If a ball is thrown upward from the top of a building 128 ft high with an initial velocity of 16 ft/s, then the height h above the ground t seconds later will be

$$h = 128 + 16t - 16t^2$$

During what time interval will the ball be at least 32 ft above the ground?

147–148 Solve the equation for x.

147. $|x + 3| = |2x + 1|$ **148.** $|3x + 5| = 1$

149–156 Solve the inequality.

149. $|x| < 3$ **150.** $|x| \geqslant 3$

151. $|x - 4| < 1$ **152.** $|x - 6| < 0.1$

153. $|x + 5| \geqslant 2$ **154.** $|x + 1| \geqslant 3$

155. $|2x - 3| \leqslant 0.4$ **156.** $|5x - 2| < 6$

157. Solve the inequality $a(bx - c) \geqslant bc$ for x, assuming that a, b, and c are positive constants.

158. Solve the inequality $ax + b < c$ for x, assuming that a, b, and c are negative constants.

159 Prove that $|ab| = |a|\,|b|$. [*Hint:* Use Equation 3.]

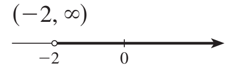
160. Show that if $0 < a < b$, then $a^2 < b^2$.

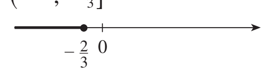
ANSWERS

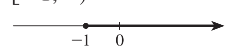
S Click here for solutions.

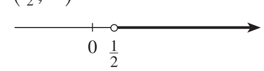
1. $-3a^2bc$ **2.** $2x^3y^5$ **3.** $2x^2 - 10x$ **4.** $4x - 3x^2$
5. $-8 + 6a$ **6.** $4 - x$ **7.** $-x^2 + 6x + 3$
8. $-3t^2 + 21t - 22$ **9.** $12x^2 + 25x - 7$
10. $x^3 + x^2 - 2x$ **11.** $4x^2 - 4x + 1$
12. $9x^2 + 12x + 4$ **13.** $30y^4 + y^5 - y^6$
14. $-15t^2 - 56t + 31$ **15.** $2x^3 - 5x^2 - x + 1$
16. $x^4 - 2x^3 - x^2 + 2x + 1$ **17.** $1 + 4x$ **18.** $3 - 2/b$
19. $\dfrac{3x+7}{x^2+2x-15}$ **20.** $\dfrac{2x}{x^2-1}$ **21.** $\dfrac{u^2+3u+1}{u+1}$
22. $\dfrac{2b^2-3ab+4a^2}{a^2b^2}$ **23.** $\dfrac{x}{yz}$ **24.** $\dfrac{zx}{y}$ **25.** $\dfrac{rs}{3t}$
26. $\dfrac{a^2}{b^2}$ **27.** $\dfrac{c}{c-2}$ **28.** $\dfrac{3+2x}{2+x}$ **29.** $2x(1+6x^2)$
30. $ab(5-8c)$ **31.** $(x+6)(x+1)$ **32.** $(x-3)(x+2)$
33. $(x-4)(x+2)$ **34.** $(2x-1)(x+4)$
35. $9(x-2)(x+2)$ **36.** $(4x+3)(2x+1)$
37. $(3x+2)(2x-3)$ **38.** $(x+5)^2$
39. $(t+1)(t^2-t+1)$ **40.** $(2t-3s)(2t+3s)$
41. $(2t-3)^2$ **42.** $(x-3)(x^2+3x+9)$
43. $x(x+1)^2$ **44.** $(x-1)^2(x-2)$
45. $(x-1)(x+1)(x+3)$ **46.** $(x-3)(x+5)(x-4)$
47. $(x-2)(x+3)(x+4)$ **48.** $(x-2)(x-3)(x+2)$
49. $\dfrac{x+2}{x-2}$ **50.** $\dfrac{2x+1}{x+2}$ **51.** $\dfrac{x+1}{x-8}$ **52.** $\dfrac{x(x+2)}{x-4}$
53. $\dfrac{x-2}{x^2-9}$ **54.** $\dfrac{x^2-6x-4}{(x-1)(x+2)(x-4)}$
55. $(x+1)^2 + 4$ **56.** $(x-8)^2 + 16$ **57.** $\left(x - \frac{5}{2}\right)^2 + \frac{15}{4}$
58. $\left(x + \frac{3}{2}\right)^2 - \frac{5}{4}$ **59.** $(2x+1)^2 - 3$
60. $3(x-4)^2 + 2$ **61.** $1, -10$ **62.** $-2, 4$
63. $\dfrac{-9 \pm \sqrt{85}}{2}$ **64.** $1 \pm 2\sqrt{2}$ **65.** $\dfrac{-5 \pm \sqrt{13}}{6}$
66. $\dfrac{-7 \pm \sqrt{33}}{4}$ **67.** $1, \dfrac{-1 \pm \sqrt{5}}{2}$ **68.** $-1, -1 \pm \sqrt{2}$
69. Irreducible **70.** Not irreducible
71. Not irreducible (two real roots) **72.** Irreducible
73. $a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$
74. $a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$
75. $x^8 - 4x^6 + 6x^4 - 4x^2 + 1$
76. $243 + 405x^2 + 270x^4 + 90x^6 + 15x^8 + x^{10}$
77. 8 **78.** $-\frac{1}{3}$ **79.** $2|x|$ **80.** $x^2|y|$
81. $4a^2b\sqrt{b}$ **82.** $2a$ **83.** 3^{26} **84.** 2^{60} **85.** $16x^{10}$
86. a^{2n+3} **87.** $\dfrac{a^2}{b}$ **88.** $\dfrac{(x+y)^2}{xy}$ **89.** $\dfrac{1}{\sqrt{3}}$
90. $2^5\sqrt{3}$ **91.** 25 **92.** $\frac{1}{256}$ **93.** $2\sqrt{2}\,|x|^3y^6$
94. $\dfrac{x^3}{y^{9/5}z^6}$ **95.** $y^{6/5}$ **96.** $a^{3/4}$ **97.** $t^{-5/2}$ **98.** $\dfrac{1}{x^{1/8}}$

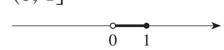
99. $\dfrac{t^{1/4}}{s^{1/24}}$ **100.** $r^{n/2}$ **101.** $\dfrac{1}{\sqrt{x}+3}$ **102.** $\dfrac{-1}{\sqrt{x}+x}$
103. $\dfrac{x^2+4x+16}{x\sqrt{x}+8}$ **104.** $\dfrac{2}{\sqrt{2+h}-\sqrt{2-h}}$
105. $\dfrac{3+\sqrt{5}}{2}$ **106.** $\dfrac{\sqrt{x}+\sqrt{y}}{x-y}$ **107.** $\dfrac{3x+4}{\sqrt{x^2+3x+4}+x}$
108. $\dfrac{2x}{\sqrt{x^2+x}+\sqrt{x^2-x}}$ **109.** False **110.** False
111. True **112.** False **113.** False **114.** False
115. False **116.** True **117.** 18 **118.** $\pi - 2$
119. $5 - \sqrt{5}$ **120.** 1 **121.** $2 - x$ **122.** $x - 2$
123. $|x+1| = \begin{cases} x+1 & \text{if } x \geqslant -1 \\ -x-1 & \text{if } x < -1 \end{cases}$
124. $|2x-1| = \begin{cases} 2x-1 & \text{if } x \geqslant \frac{1}{2} \\ 1-2x & \text{if } x < \frac{1}{2} \end{cases}$
125. $x^2 + 1$
126. $|1-2x^2| = \begin{cases} 1-2x^2 & \text{if } -1/\sqrt{2} \leqslant x \leqslant 1/\sqrt{2} \\ 2x^2-1 & \text{if } x < -1/\sqrt{2} \text{ or } x > 1/\sqrt{2} \end{cases}$

127. $(-2, \infty)$


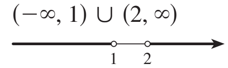
128. $\left(-\infty, -\frac{2}{3}\right]$


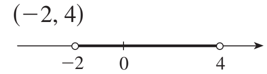
129. $[-1, \infty)$


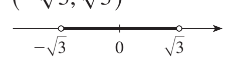
130. $\left(\frac{1}{2}, \infty\right)$


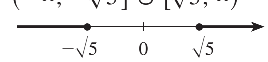
131. $(0, 1]$


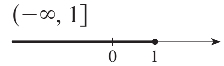
132. $(-1, 4]$

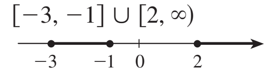

133. $(-\infty, 1) \cup (2, \infty)$


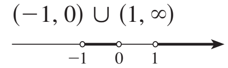
134. $(-2, 4)$


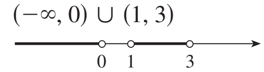
135. $(-\sqrt{3}, \sqrt{3})$


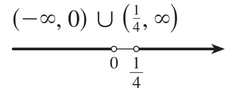
136. $\left(-\infty, -\sqrt{5}\right] \cup \left[\sqrt{5}, \infty\right)$


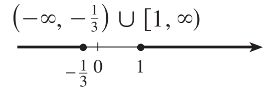
137. $(-\infty, 1]$


138. $[-3, -1] \cup [2, \infty)$


139. $(-1, 0) \cup (1, \infty)$


140. $(-\infty, 0) \cup (1, 3)$


141. $(-\infty, 0) \cup \left(\frac{1}{4}, \infty\right)$


142. $\left(-\infty, -\frac{1}{3}\right) \cup [1, \infty)$


143. $[10, 35]$ **144.** $[68, 86]$
145. (a) $T = 20 - 10h,\ 0 \leqslant h \leqslant 12$ (b) $-30°\text{C} \leqslant T \leqslant 20°\text{C}$
146. $[0, 3]$ **147.** $2, -\frac{4}{3}$ **148.** $-\frac{4}{3}, -2$ **149.** $(-3, 3)$
150. $(-\infty, -3] \cup [3, \infty)$ **151.** $(3, 5)$ **152.** $(5.9, 6.1)$
153. $(-\infty, -7] \cup [-3, \infty)$ **154.** $(-\infty, -4] \cup [2, \infty)$
155. $[1.3, 1.7]$ **156.** $\left(-\frac{4}{5}, \frac{8}{5}\right)$
157. $x \geqslant \dfrac{(a+b)c}{ab}$ **158.** $x > \dfrac{c-b}{a}$

SOLUTIONS

1. $(-6ab)(0.5ac) = (-6)(0.5)(a \cdot abc) = -3a^2bc$

2. $-(2x^2y)(-xy^4) = 2x^2xyy^4 = 2x^3y^5$

3. $2x(x-5) = 2x \cdot x - 2x \cdot 5 = 2x^2 - 10x$

4. $(4-3x)x = 4 \cdot x - 3x \cdot x = 4x - 3x^2$

5. $-2(4-3a) = -2 \cdot 4 + 2 \cdot 3a = -8 + 6a$

6. $8 - (4+x) = 8 - 4 - x = 4 - x$

7. $$\begin{aligned} 4(x^2 - x + 2) - 5(x^2 - 2x + 1) &= 4x^2 - 4x + 8 - 5x^2 - 5(-2x) - 5 \\ &= 4x^2 - 5x^2 - 4x + 10x + 8 - 5 = -x^2 + 6x + 3 \end{aligned}$$

8. $$\begin{aligned} 5(3t-4) - (t^2+2) - 2t(t-3) &= 15t - 20 - t^2 - 2 - 2t^2 + 6t \\ &= (-1-2)t^2 + (15+6)t - 20 - 2 = -3t^2 + 21t - 22 \end{aligned}$$

9. $(4x-1)(3x+7) = 4x(3x+7) - (3x+7) = 12x^2 + 28x - 3x - 7 = 12x^2 + 25x - 7$

10. $$\begin{aligned} x(x-1)(x+2) &= (x^2 - x)(x+2) = x^2(x+2) - x(x+2) = x^3 + 2x^2 - x^2 - 2x \\ &= x^3 + x^2 - 2x \end{aligned}$$

11. $(2x-1)^2 = (2x)^2 - 2(2x)(1) + 1^2 = 4x^2 - 4x + 1$

12. $(2+3x)^2 = 2^2 + 2(2)(3x) + (3x)^2 = 9x^2 + 12x + 4$

13. $$\begin{aligned} y^4(6-y)(5+y) &= y^4[6(5+y) - y(5+y)] = y^4(30 + 6y - 5y - y^2) \\ &= y^4(30 + y - y^2) = 30y^4 + y^5 - y^6 \end{aligned}$$

14. $$\begin{aligned} (t-5)^2 - 2(t+3)(8t-1) &= t^2 - 2(5t) + 5^2 - 2(8t^2 - t + 24t - 3) \\ &= t^2 - 10t + 25 - 16t^2 + 2t - 48t + 6 = -15t^2 - 56t + 31 \end{aligned}$$

15. $$\begin{aligned} (1+2x)(x^2 - 3x + 1) &= 1(x^2 - 3x + 1) + 2x(x^2 - 3x + 1) = x^2 - 3x + 1 + 2x^3 - 6x^2 + 2x \\ &= 2x^3 - 5x^2 - x + 1 \end{aligned}$$

16. $$\begin{aligned} (1 + x - x^2)^2 &= (1 + x - x^2)(1 + x - x^2) = 1(1 + x - x^2) + x(1 + x - x^2) - x^2(1 + x - x^2) \\ &= 1 + x - x^2 + x + x^2 - x^3 - x^2 - x^3 + x^4 = x^4 - 2x^3 - x^2 + 2x + 1 \end{aligned}$$

17. $\dfrac{2+8x}{2} = \dfrac{2}{2} + \dfrac{8x}{2} = 1 + 4x$

18. $\dfrac{9b-6}{3b} = \dfrac{9b}{3b} - \dfrac{6}{3b} = 3 - \dfrac{2}{b}$

19. $\dfrac{1}{x+5} + \dfrac{2}{x-3} = \dfrac{(1)(x-3) + 2(x+5)}{(x+5)(x-3)} = \dfrac{x - 3 + 2x + 10}{(x+5)(x-3)} = \dfrac{3x+7}{x^2 + 2x - 15}$

20. $\dfrac{1}{x+1} + \dfrac{1}{x-1} = \dfrac{1(x-1) + 1(x+1)}{(x+1)(x-1)} = \dfrac{x - 1 + x + 1}{x^2 - 1} = \dfrac{2x}{x^2 - 1}$

21. $u + 1 + \dfrac{u}{u+1} = \dfrac{(u+1)(u+1) + u}{u+1} = \dfrac{u^2 + 2u + 1 + u}{u+1} = \dfrac{u^2 + 3u + 1}{u+1}$

22. $\dfrac{2}{a^2} - \dfrac{3}{ab} + \dfrac{4}{b^2} = \dfrac{2b^2}{a^2b^2} - \dfrac{3ab}{a^2b^2} + \dfrac{4a^2}{a^2b^2} = \dfrac{2b^2 - 3ab + 4a^2}{a^2b^2}$

23. $\dfrac{x/y}{z} = \dfrac{x/y}{z/1} = \dfrac{1}{z} \cdot \dfrac{x}{y} = \dfrac{x}{yz}$

24. $\dfrac{x}{y/z} = \dfrac{x/1}{y/z} = \dfrac{z}{y} \cdot \dfrac{x}{1} = \dfrac{zx}{y}$

25. $\left(\dfrac{-2r}{s}\right)\left(\dfrac{s^2}{-6t}\right) = \dfrac{-2rs^2}{-6st} = \dfrac{rs}{3t}$

26. $\dfrac{a}{bc} \div \dfrac{b}{ac} = \dfrac{a}{bc} \times \dfrac{ac}{b} = \dfrac{a^2c}{b^2c} = \dfrac{a^2}{b^2}$

27. $\dfrac{1+\dfrac{1}{c-1}}{1-\dfrac{1}{c-1}} = \dfrac{\dfrac{c-1+1}{c-1}}{\dfrac{c-1-1}{c-1}} = \dfrac{\dfrac{c}{c-1}}{\dfrac{c-2}{c-1}} = \dfrac{c-1}{c-2}\cdot\dfrac{c}{c-1} = \dfrac{c}{c-2}$

28. $1+\dfrac{1}{1+\dfrac{1}{1+x}} = 1+\dfrac{1}{\dfrac{1+x+1}{1+x}} = 1+\dfrac{1+x}{2+x} = \dfrac{2+x+1+x}{2+x} = \dfrac{3+2x}{2+x}$

29. $2x + 12x^3 = 2x\cdot 1 + 2x\cdot 6x^2 = 2x(1+6x^2)$

30. $5ab - 8abc = ab\cdot 5 - ab\cdot 8c = ab(5-8c)$

31. The two integers that add to give 7 and multiply to give 6 are 6 and 1. Therefore $x^2+7x+6=(x+6)(x+1)$.

32. The two integers that add to give -1 and multiply to give -6 are -3 and 2.
Therefore $x^2-2x-6=(x-3)(x+2)$.

33. The two integers that add to give -2 and multiply to give -8 are -4 and 2.
Therefore $x^2-2x-8=(x-4)(x+2)$.

34. $2x^2+7x-4=(2x-1)(x+4)$

35. $9x^2-36=9(x^2-4)=9(x-2)(x+2)$ [Equation 3 with $a=x$, $b=2$]

36. $8x^2+10x+3=(4x+3)(2x+1)$

37. $6x^2-5x-6=(3x+2)(2x-3)$

38. $x^2+10x+25=(x+5)^2$ [Equation 1 with $a-x$, $b=5$]

39. $t^3+1=(t+1)(t^2-t+1)$ [Equation 5 with $a=t$, $b=1$]

40. $4t^2-9s^2=(2t)^2-(3s)^2=(2t-3s)(2t+3s)$ [Equation 3 with $a=2t$, $b=3s$]

41. $4t^2-12t+9=(2t-3)^2$ [Equation 2 with $a=2t$, $b=3$]

42. $x^3-27=(x-3)(x^2+3x+9)$ [Equation 4 with $a=x$, $b=3$]

43. $x^3+2x^2+x=x(x^2+2x+1)=x(x+1)^2$ [Equation 1 with $a=x$, $b=1$]

44. Let $p(x)=x^3-4x^2+5x-2$, and notice that $p(1)=0$, so by the Factor Theorem, $(x-1)$ is a factor.
Use long division (as in Example 8):

$$\begin{array}{r|l}
 & x^2 - 3x + 2 \\ \cline{2-2}
x-1 & x^3 - 4x^2 + 5x - 2 \\
 & \underline{x^3 - x^2} \\
 & \quad -3x^2 + 5x \\
 & \quad \underline{-3x^2 + 3x} \\
 & \qquad\qquad 2x - 2 \\
 & \qquad\qquad \underline{2x - 2}
\end{array}$$

Therefore $x^3-4x^2+5x-2=(x-1)(x^2-3x+2)=(x-1)(x-2)(x-1)=(x-1)^2(x-2)$.

45. Let $p(x)=x^3+3x^2-x-3$, and notice that $p(1)=0$, so by the Factor Theorem, $(x-1)$ is a factor.
Use long division (as in Example 8):

$$\begin{array}{r|l}
 & x^2 + 4x + 3 \\ \cline{2-2}
x-1 & x^3 + 3x^2 - x - 3 \\
 & \underline{x^3 - x^2} \\
 & \quad 4x^2 - x \\
 & \quad \underline{4x^2 - 4x} \\
 & \qquad\qquad 3x - 3 \\
 & \qquad\qquad \underline{3x - 3}
\end{array}$$

Therefore $x^3+3x^2-x-3=(x-1)(x^2+4x+3)=(x-1)(x+1)(x+3)$.

46. Let $p(x) = x^3 - 2x^2 - 23x + 60$, and notice that $p(3) = 0$, so by the Factor Theorem, $(x - 3)$ is a factor. Use long division (as in Example 8):

$$\begin{array}{r|l} & x^2 + x - 20 \\ \hline x - 3 & x^3 - 2x^2 - 23x + 60 \\ & x^3 - 3x^2 \\ \hline & \quad x^2 - 23x \\ & \quad x^2 - 3x \\ \hline & \qquad -20x + 60 \\ & \qquad -20x + 60 \\ \hline \end{array}$$

Therefore $x^3 - 2x^2 - 23x + 60 = (x - 3)(x^2 + x - 20) = (x - 3)(x + 5)(x - 4)$.

47. Let $p(x) = x^3 + 5x^2 - 2x - 24$, and notice that $p(2) = 2^3 + 5(2)^2 - 2(2) - 24 = 0$, so by the Factor Theorem, $(x - 2)$ is a factor. Use long division (as in Example 8):

$$\begin{array}{r|l} & x^2 + 7x + 12 \\ \hline x - 2 & x^3 + 5x^2 - 2x - 24 \\ & x^3 - 2x^2 \\ \hline & \quad 7x^2 - 2x \\ & \quad 7x^2 - 14x \\ \hline & \qquad 12x - 24 \\ & \qquad 12x - 24 \\ \hline \end{array}$$

Therefore $x^3 + 5x^2 - 2x - 24 = (x - 2)(x^2 + 7x + 12) = (x - 2)(x + 3)(x + 4)$.

48. Let $p(x) = x^3 - 3x^2 - 4x + 12$, and notice that $p(2) = 0$, so by the Factor Theorem, $(x - 2)$ is a factor. Use long division (as in Example 8):

$$\begin{array}{r|l} & x^2 - x - 6 \\ \hline x - 2 & x^3 - 3x^2 - 4x + 12 \\ & x^3 - 2x^2 \\ \hline & \quad -x^2 - 4x \\ & \quad -x^2 + 2x \\ \hline & \qquad -6x + 12 \\ & \qquad -6x + 12 \\ \hline \end{array}$$

Therefore $x^3 - 3x^2 - 4x + 12 = (x - 2)(x^2 - x - 6) = (x - 2)(x - 3)(x + 2)$.

49. $\dfrac{x^2 + x - 2}{x^2 - 3x + 2} = \dfrac{(x + 2)(x - 1)}{(x - 2)(x - 1)} = \dfrac{x + 2}{x - 2}$

50. $\dfrac{2x^2 - 3x - 2}{x^2 - 4} = \dfrac{(2x + 1)(x - 2)}{(x - 2)(x + 2)} = \dfrac{2x + 1}{x + 2}$

51. $\dfrac{x^2 - 1}{x^2 - 9x + 8} = \dfrac{(x - 1)(x + 1)}{(x - 8)(x - 1)} = \dfrac{x + 1}{x - 8}$

52. $\dfrac{x^3 + 5x^2 + 6x}{x^2 - x - 12} = \dfrac{x(x^2 + 5x + 6)}{(x - 4)(x + 3)} = \dfrac{x(x + 3)(x + 2)}{(x - 4)(x + 3)} = \dfrac{x(x + 2)}{x - 4}$

53. $\dfrac{1}{x + 3} + \dfrac{1}{x^2 - 9} = \dfrac{1}{x + 3} + \dfrac{1}{(x - 3)(x + 3)} - \dfrac{1(x - 3) + 1}{(x - 3)(x + 3)} = \dfrac{x - 2}{x^2 - 9}$

54. $\dfrac{x}{x^2 + x - 2} - \dfrac{2}{x^2 - 5x + 4} = \dfrac{x}{(x - 1)(x + 2)} - \dfrac{2}{(x - 4)(x - 1)} = \dfrac{x(x - 4) - 2(x + 2)}{(x - 1)(x + 2)(x - 4)}$

$$= \frac{x^2 - 4x - 2x - 4}{(x - 1)(x + 2)(x - 4)} = \frac{x^2 - 6x - 4}{(x - 1)(x + 2)(x - 4)}$$

55. $x^2 + 2x + 5 = [x^2 + 2x] + 5 = [x^2 + 2x + (1)^2 - (1)^2] + 5 = (x + 1)^2 + 5 - 1 = (x + 1)^2 + 4$

56. $x^2-16x+80=[x^2-16x]+80=[x^2-16x+(8)^2-(8)^2]+80=(x-8)^2+80-64=(x-8)^2+16$

57. $x^2-5x+10=[x^2-5x]+10=\left[x^2-5x+\left(-\frac{5}{2}\right)^2-\left(-\frac{5}{2}\right)^2\right]+10=\left(x-\frac{5}{2}\right)^2+10-\frac{25}{4}=\left(x-\frac{5}{2}\right)^2+\frac{15}{4}$

58. $x^2+3x+1=[x^2+3x]+1=\left[x^2+3x+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2\right]+1=\left(x+\frac{3}{2}\right)^2+1-\left(\frac{3}{2}\right)^2=\left(x+\frac{3}{2}\right)^2-\frac{5}{4}$

59. $4x^2+4x-2=4[x^2+x]-2=4\left[x^2+x+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2\right]-2=4\left(x+\frac{1}{2}\right)^2-2-4\left(\frac{1}{4}\right)=4\left(x+\frac{1}{2}\right)^2-3$

60. $3x^2-24x+50=3[x^2-8x]+50=3[x^2-8x+(-4)^2-(-4)^2]+50=3(x-4)^2+50-3(-4)^2$
$=3(x-4)^2+2$

61. $x^2-9x-10=0 \quad\Leftrightarrow\quad (x+10)(x-1)=0 \quad\Leftrightarrow\quad x+10=0$ or $x-1=0 \quad\Leftrightarrow\quad x=-10$ or $x=1$.

62. $x^2-2x-8=0 \quad\Leftrightarrow\quad (x-4)(x+2)=0 \quad\Leftrightarrow\quad x-4=0$ or $x+2=0 \quad\Leftrightarrow\quad x=4$ or $x=-2$.

63. Using the quadratic formula, $x^2+9x-1=0 \quad\Leftrightarrow\quad x=\dfrac{-9\pm\sqrt{9^2-4(1)(-1)}}{2(1)}=\dfrac{9\pm\sqrt{85}}{2}$.

64. Using the quadratic formula, $x^2-2x-7=0 \quad\Leftrightarrow\quad x=\dfrac{2\pm\sqrt{4-4(1)(-7)}}{2}=\dfrac{2\pm\sqrt{32}}{2}=1\pm2\sqrt{2}$.

65. Using the quadratic formula, $3x^2+5x+1=0 \quad\Leftrightarrow\quad x=\dfrac{-5\pm\sqrt{5^2-4(3)(1)}}{2(3)}=\dfrac{-5\pm\sqrt{13}}{6}$.

66. Using the quadratic formula, $2x^2+7x+2=0 \quad\Leftrightarrow\quad x=\dfrac{-7\pm\sqrt{49-4(2)(2)}}{2(2)}=\dfrac{-7\pm\sqrt{33}}{4}$.

67. Let $p(x)=x^3-2x+1$, and notice that $p(1)=0$, so by the Factor Theorem, $(x-1)$ is a factor.
Use long division:

$$\begin{array}{r|l}
 & x^2 + x - 1 \\ \hline
x-1 & x^3 + 0x^2 - 2x + 1 \\
 & x^3 - x^2 \\ \hline
 & \quad x^2 - 2x \\
 & \quad x^2 - x \\ \hline
 & \qquad -x + 1 \\
 & \qquad -x + 1 \\ \hline
\end{array}$$

Therefore $x^3-2x+1=(x-1)(x^2+x-1)=0 \quad\Leftrightarrow\quad x-1=0$ or $x^2+x-1=0 \quad\Leftrightarrow$

$x=1$ or [using the quadratic formula] $x=\dfrac{-1\pm\sqrt{1^2-4(1)(-1)}}{2(1)}=\dfrac{-1\pm\sqrt{5}}{2}$.

68. Let $p(x)=x^3+3x^2+x-1$, and notice that $p(-1)=0$, so by the Factor Theorem, $(x+1)$ is a factor.
Use long division:

$$\begin{array}{r|l}
 & x^2 + 2x - 1 \\ \hline
x+1 & x^3 + 3x^2 + x - 1 \\
 & x^3 + x^2 \\ \hline
 & \quad 2x^2 + x \\
 & \quad 2x^2 + 2x \\ \hline
 & \qquad -x - 1 \\
 & \qquad -x - 1 \\ \hline
\end{array}$$

Therefore $x^3+3x^2+x-1=(x+1)(x^2+2x-1)=0 \quad\Leftrightarrow\quad x+1=0$ or $x^2+2x-1=0 \quad\Leftrightarrow$

$x=-1$ or [using the quadratic formula] $x=\dfrac{-2\pm\sqrt{2^2-4(1)(-1)}}{2}=-1\pm\sqrt{2}$.

69. $2x^2 + 3x + 4$ is irreducible because its discriminant is negative: $b^2 - 4ac = 9 - 4(2)(4) = -23 < 0$.

70. The quadratic $2x^2 + 9x + 4$ is not irreducible because $b^2 - 4ac = 9^2 - 4(2)(4) = 49 > 0$.

71. $3x^2 + x - 6$ is not irreducible because its discriminant is nonnegative: $b^2 - 4ac = 1 - 4(3)(-6) = 73 > 0$.

72. The quadratic $x^2 + 3x + 6$ is irreducible because $b^2 - 4ac = 3^2 - 4(1)(6) = -15 < 0$.

73. Using the Binomial Theorem with $k = 6$ we have

$$\begin{aligned}(a+b)^6 &= a^6 + 6a^5b + \frac{6\cdot 5}{1\cdot 2}a^4b^2 + \frac{6\cdot 5\cdot 4}{1\cdot 2\cdot 3}a^3b^3 + \frac{6\cdot 5\cdot 4\cdot 3}{1\cdot 2\cdot 3\cdot 4}a^2b^4 + 6ab^5 + b^6\\ &= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6\end{aligned}$$

74. Using the Binomial Theorem with $k = 7$ we have

$$\begin{aligned}(a+b)^7 &= a^7 + 7a^6b + \frac{7\cdot 6}{1\cdot 2}a^5b^2 + \frac{7\cdot 6\cdot 5}{1\cdot 2\cdot 3}a^4b^3 + \frac{7\cdot 6\cdot 5\cdot 4}{1\cdot 2\cdot 3\cdot 4}a^3b^4 + \frac{7\cdot 6\cdot 5\cdot 4\cdot 3}{1\cdot 2\cdot 3\cdot 4\cdot 5}a^2b^5 + 7ab^6 + b^7\\ &= a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7\end{aligned}$$

75. Using the Binomial Theorem with $a = x^2$, $b = -1$, $k = 4$ we have

$$\begin{aligned}(x^2-1)^4 &= [x^2 + (-1)]^4 = (x^2)^4 + 4(x^2)^3(-1) + \frac{4\cdot 3}{1\cdot 2}(x^2)^2(-1)^2 + 4(x^2)(-1)^3 + (-1)^4\\ &= x^8 - 4x^6 + 6x^4 - 4x^2 + 1\end{aligned}$$

76. Using the Binomial Theorem with $a = 3$, $b = x^2$, $k = 5$ we have

$$\begin{aligned}(3+x^2)^5 &= 3^5 + 5(3)^4(x^2)^1 + \frac{5\cdot 4}{1\cdot 2}(3)^3(x^2)^2 + \frac{5\cdot 4\cdot 3}{1\cdot 2\cdot 3}(3)^2(x^2)^3 + 5(3)(x^2)^4 + (x^2)^5\\ &= 243 + 405x^2 + 270x^4 + 90x^6 + 15x^8 + x^{10}\end{aligned}$$

77. Using Equation 10, $\sqrt{32}\,\sqrt{2} = \sqrt{32\cdot 2} = \sqrt{64} = 8$.

78. $\dfrac{\sqrt[3]{-2}}{\sqrt[3]{54}} = \sqrt[3]{\dfrac{-2}{54}} = \sqrt[3]{\dfrac{-1}{27}} = \dfrac{\sqrt[3]{-1}}{\sqrt[3]{27}} = \dfrac{-1}{3} = -\dfrac{1}{3}$

79. Using Equation 10, $\dfrac{\sqrt[4]{32x^4}}{\sqrt[4]{2}} = \dfrac{\sqrt[4]{32}\,\sqrt[4]{x^4}}{\sqrt[4]{2}} = \sqrt[4]{\dfrac{32}{2}}\,\sqrt[4]{x^4} = \sqrt[4]{16}\,|x| = 2\,|x|$.

80. $\sqrt{xy}\sqrt{x^3y} = \sqrt{(xy)(x^3y)} = \sqrt{x^4y^2} = x^2\,|y|$

81. Using Equation 10, $\sqrt{16a^4b^3} = \sqrt{16}\sqrt{a^4}\sqrt{b^3} = 4a^2b^{3/2} = 4a^2b\,b^{1/2} = 4a^2b\sqrt{b}$.

82. $\dfrac{\sqrt[5]{96a^6}}{\sqrt[5]{3a}} = \sqrt[5]{\dfrac{96a^6}{3a}} = \sqrt[5]{32a^5} = 2a$

83. Using Laws 3 and 1 of Exponents respectively, $3^{10}\times 9^8 = 3^{10}\times(3^2)^8 = 3^{10}\times 3^{2\cdot 8} = 3^{10+16} = 3^{26}$.

84. Using Laws 3 and 1, $2^{16}\times 4^{10}\times 16^6 = 2^{16}\times(2^2)^{10}\times(2^4)^6 = 2^{16}\times 2^{20}\times 2^{24} = 2^{60}$.

85. Using Laws 4, 1, and 2 of Exponents respectively, $\dfrac{x^9(2x)^4}{x^3} = \dfrac{x^9(2^4)x^4}{x^3} = \dfrac{16x^{9+4}}{x^3} = 16x^{9+4-3} = 16x^{10}$.

86. Using Laws 1 and 2, $\dfrac{a^n\times a^{2n+1}}{a^{n-2}} = \dfrac{a^{n+2n+1}}{a^{n-2}} = \dfrac{a^{3n+1}}{a^{n-2}} = a^{3n+1-(n-2)} = a^{2n+3}$.

87. Using Law 2 of Exponents, $\dfrac{a^{-3}b^4}{a^{-5}b^5} = a^{-3-(-5)}b^{4-5} = a^2b^{-1} = \dfrac{a^2}{b}$.

88. $\dfrac{x^{-1}+y^{-1}}{(x+y)^{-1}} = (x+y)\left(\dfrac{1}{x}+\dfrac{1}{y}\right) = (x+y)\left(\dfrac{y+x}{xy}\right) = \dfrac{(y+x)^2}{xy}$

89. By definitions 3 and 4 for exponents respectively, $3^{-1/2} = \dfrac{1}{3^{1/2}} = \dfrac{1}{\sqrt{3}}$.

90. $96^{1/5} = \sqrt[5]{96} = \sqrt[5]{32\cdot 3} = \sqrt[5]{32}\,\sqrt[5]{3} = 2\,\sqrt[5]{3}$

91. Using definition 4 for exponents, $125^{2/3} = \left[\sqrt[3]{125}\right]^2 = 5^2 = 25$.

92. $64^{-4/3} = \dfrac{1}{64^{4/3}} = \dfrac{1}{\left[\sqrt[3]{64}\right]^4} = \dfrac{1}{4^4} = \dfrac{1}{256}$

93. $(2x^2y^4)^{3/2} = 2^{3/2}(x^2)^{3/2}(y^4)^{3/2} = 2 \cdot 2^{1/2}\left[\sqrt{x^2}\right]^3\left[\sqrt{y^4}\right]^3 = 2\sqrt{2}\,|x|^3\,(y^2)^3 = 2\sqrt{2}\,|x|^3\,y^6$

94. $(x^{-5}y^3z^{10})^{-3/5} = (x^{-5})^{-3/5}(y^3)^{-3/5}(z^{10})^{-3/5} = x^{15/5}y^{-9/5}z^{-30/5} = \dfrac{x^2}{y^{9/5}z^6}$

95. $\sqrt[5]{y^6} = y^{6/5}$ by definition 4 for exponents.

96. $\left(\sqrt[4]{a}\right)^3 = (a^{1/4})^3 = a^{3/4}$

97. $\dfrac{1}{\left(\sqrt{t}\right)^5} = \dfrac{1}{(t^{1/2})^5} = \dfrac{1}{t^{5/2}} = t^{-5/2}$

98. $\dfrac{\sqrt[8]{x^5}}{\sqrt[4]{x^3}} = \dfrac{x^{5/8}}{x^{3/4}} = x^{(5/8)\,-\,(3/4)} = x^{-1/8} = \dfrac{1}{x^{1/8}}$

99. $\sqrt[4]{\dfrac{t^{1/2}\sqrt{st}}{s^{2/3}}} = \left(\dfrac{t^{1/2}s^{1/2}t^{1/2}}{s^{2/3}}\right)^{1/4} = \left(t^{(1/2)\,+\,(1/2)}s^{(1/2)\,-\,(2/3)}\right)^{1/4} = (ts^{-1/6})^{1/4}$

$$= t^{1/4}s^{(-1/6)\,\cdot\,(1/4)} = \frac{t^{1/4}}{s^{1/24}}$$

100. $\sqrt[4]{r^{2n+1}} \times \sqrt[4]{r^{-1}} = \sqrt[4]{r^{2n+1} \times r^{-1}} = \sqrt[4]{r^{2n+1-1}} = \sqrt[4]{r^{2n}} = (r^{2n})^{1/4} = r^{2n/4} = r^{n/2}$

101. $\dfrac{\sqrt{x}-3}{x-9} = \dfrac{\sqrt{x}-3}{x-9} \cdot \dfrac{\sqrt{x}+3}{\sqrt{x}+3} = \dfrac{(x-9)}{(x-9)\left(\sqrt{x}+3\right)} = \dfrac{1}{\sqrt{x}+3}$

102. $\dfrac{\frac{1}{\sqrt{x}}-1}{x-1} = \dfrac{\frac{1}{\sqrt{x}}-1}{x-1} \cdot \dfrac{\frac{1}{\sqrt{x}}+1}{\frac{1}{\sqrt{x}}+1} = \dfrac{\frac{1}{x}-1}{(x-1)\left(\frac{1}{\sqrt{x}}+1\right)} = \dfrac{\frac{1-x}{x}}{(x-1)\left(\frac{1}{\sqrt{x}}+1\right)} = \dfrac{-1}{x\left(\frac{1}{\sqrt{x}}+1\right)} = \dfrac{-1}{\sqrt{x}+x}$

103. $\dfrac{x\sqrt{x}-8}{x-4} = \dfrac{x\sqrt{x}-8}{x-4} \cdot \dfrac{x\sqrt{x}+8}{x\sqrt{x}+8} = \dfrac{x^3-64}{(x-4)(x\sqrt{x}+8)}$

$$= \frac{(x-4)(x^2+4x+16)}{(x-4)(x\sqrt{x}+8)} \quad [\text{Equation 4 with } a = x,\ b = 4] \quad = \frac{x^2+4x+16}{x\sqrt{x}+8}$$

104. $\dfrac{\sqrt{2+h}+\sqrt{2-h}}{h} = \dfrac{\sqrt{2+h}+\sqrt{2-h}}{h} \cdot \dfrac{\sqrt{2+h}-\sqrt{2-h}}{\sqrt{2+h}-\sqrt{2-h}} = \dfrac{2+h-(2-h)}{h\left(\sqrt{2+h}-\sqrt{2-h}\right)}$

$$= \frac{2}{\sqrt{2+h}-\sqrt{2-h}}$$

105. $\dfrac{2}{3-\sqrt{5}} = \dfrac{2}{3-\sqrt{5}} \cdot \dfrac{3+\sqrt{5}}{3+\sqrt{5}} = \dfrac{2\left(3+\sqrt{5}\right)}{9-5} = \dfrac{3+\sqrt{5}}{2}$

106. $\dfrac{1}{\sqrt{x}-\sqrt{y}} = \dfrac{1}{\sqrt{x}-\sqrt{y}} \cdot \dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}} = \dfrac{\sqrt{x}+\sqrt{y}}{x-y}$

107. $\sqrt{x^2+3x+4}-x = \left(\sqrt{x^2+3x+4}-x\right) \cdot \dfrac{\sqrt{x^2+3x+4}+x}{\sqrt{x^2+3x+4}+x} = \dfrac{x^2+3x+4-x^2}{\sqrt{x^2+3x+4}+x} = \dfrac{3x+4}{\sqrt{x^2+3x+4}+x}$

108. $\sqrt{x^2+x}-\sqrt{x^2-x} = \left(\sqrt{x^2+x}-\sqrt{x^2-x}\right) \cdot \dfrac{\sqrt{x^2+x}+\sqrt{x^2-x}}{\sqrt{x^2+x}+\sqrt{x^2-x}} = \dfrac{x^2+x-(x^2-x)}{\sqrt{x^2+x}+\sqrt{x^2-x}}$

$$= \frac{2x}{\sqrt{x^2+x}+\sqrt{x^2-x}}$$

109. False. See Example 14(b).

110. False. See the warning after Equation 10.

111. True: $\dfrac{16+a}{16} = \dfrac{16}{16} + \dfrac{a}{16} = 1 + \dfrac{a}{16}$

112. False: $\dfrac{1}{x^{-1}+y^{-1}} = \dfrac{1}{\frac{1}{x}+\frac{1}{y}} = \dfrac{1}{\frac{x+y}{xy}} = \dfrac{xy}{x+y} \neq x+y$

113. False.

114. False. See the warning on page 2.

115. False. Using Law 3 of Exponents, $(x^3)^4 = x^{3\cdot 4} = x^{12} \neq x^7$.

116. True.

117. $|5-23| = |-18| = 18$

118. $|\pi - 2| = \pi - 2$ because $\pi - 2 > 0$.

119. $\left|\sqrt{5}-5\right| = -\left(\sqrt{5}-5\right) = 5-\sqrt{5}$ because $\sqrt{5}-5<0$.

120. $\Big||-2|-|-3|\Big| = |2-3| = |-1| = 1$

121. If $x<2$, $x-2<0$, so $|x-2| = -(x-2) = 2-x$.

122. If $x>2$, $x-2>0$, so $|x-2| = x-2$.

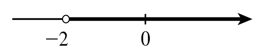
123. $|x+1| = \begin{cases} x+1 & \text{if } x+1 \geq 0 \\ -(x+1) & \text{if } x+1<0 \end{cases} = \begin{cases} x+1 & \text{if } x \geq -1 \\ -x-1 & \text{if } x<-1 \end{cases}$

124. $|2x-1| = \begin{cases} 2x-1 & \text{if } 2x-1 \geq 0 \\ -(2x-1) & \text{if } 2x-1<0 \end{cases} = \begin{cases} 2x-1 & \text{if } x \geq \frac{1}{2} \\ 1-2x & \text{if } x<\frac{1}{2} \end{cases}$

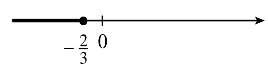
125. $\left|x^2+1\right| = x^2+1$ (since $x^2+1 \geq 0$ for all x).

126. Determine when $1-2x^2<0 \quad\Leftrightarrow\quad 1<2x^2 \quad\Leftrightarrow\quad x^2>\frac{1}{2} \quad\Leftrightarrow\quad \sqrt{x^2}>\sqrt{\frac{1}{2}} \quad\Leftrightarrow\quad |x|>\sqrt{\frac{1}{2}} \quad\Leftrightarrow$ $x<-\frac{1}{\sqrt{2}}$ or $x>\frac{1}{\sqrt{2}}$. Thus, $\left|1-2x^2\right| = \begin{cases} 1-2x^2 & \text{if } -\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}} \\ 2x^2-1 & \text{if } x<-\frac{1}{\sqrt{2}} \text{ or } x>\frac{1}{\sqrt{2}} \end{cases}$

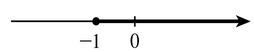
127. $2x+7>3 \quad\Leftrightarrow\quad 2x>-4 \quad\Leftrightarrow\quad x>-2$, so $x \in (-2,\infty)$.



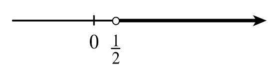
128. $4-3x \geq 6 \quad\Leftrightarrow\quad -3x \geq 2 \quad\Leftrightarrow\quad x \leq -\frac{2}{3}$, so $x \in \left(-\infty, -\frac{2}{3}\right]$.



129. $1-x \leq 2 \quad\Leftrightarrow\quad -x \leq 1 \quad\Leftrightarrow\quad x \geq -1$, so $x \in [-1,\infty)$.

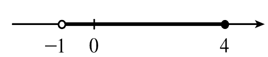


130. $1+5x>5-3x \quad\Leftrightarrow\quad 8x>4 \quad\Leftrightarrow\quad x>\frac{1}{2}$, so $x \in \left(\frac{1}{2},\infty\right)$.



131. $0 \leq 1-x<1 \quad\Leftrightarrow\quad -1 \leq -x<0 \quad\Leftrightarrow\quad 1 \geq x>0$, so $x \in (0,1]$.

132. $1<3x+4 \leq 16 \quad\Leftrightarrow\quad -3<3x \leq 12 \quad\Leftrightarrow\quad -1<x \leq 4$, so $x \in (-1,4]$.



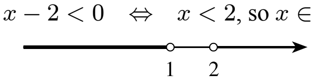
133. $(x-1)(x-2)>0$. *Case 1:* (both factors are positive, so their product is positive)

$x-1>0 \quad\Leftrightarrow\quad x>1$, and $x-2>0 \quad\Leftrightarrow\quad x>2$, so $x \in (2,\infty)$.

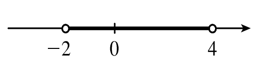
Case 2: (both factors are negative, so their product is positive)

$x-1<0 \quad\Leftrightarrow\quad x<1$, and $x-2<0 \quad\Leftrightarrow\quad x<2$, so $x \in (-\infty,1)$.

Thus, the solution set is $(-\infty,1) \cup (2,\infty)$.

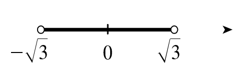


134. $x^2<2x+8 \quad\Leftrightarrow\quad x^2-2x-8<0 \quad\Leftrightarrow\quad (x-4)(x+2)<0$. *Case 1:* $x>4$ and $x<-2$, which is impossible. *Case 2:* $x<4$ and $x>-2$. Thus, the solution set is $(-2,4)$.

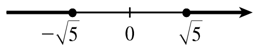


135. $x^2<3 \quad\Leftrightarrow\quad x^2-3<0 \quad\Leftrightarrow\quad \left(x-\sqrt{3}\right)\left(x+\sqrt{3}\right)<0$. *Case 1:* $x>\sqrt{3}$ and $x<-\sqrt{3}$, which is impossible. *Case 2:* $x<\sqrt{3}$ and $x>-\sqrt{3}$. Thus, the solution set is $\left(-\sqrt{3},\sqrt{3}\right)$.

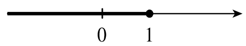
Another method: $x^2<3 \quad\Leftrightarrow\quad |x|<\sqrt{3} \quad\Leftrightarrow\quad -\sqrt{3}<x<\sqrt{3}$.



136. $x^2 \geq 5 \quad \Leftrightarrow \quad x^2 - 5 \geq 0 \quad \Leftrightarrow \quad \left(x - \sqrt{5}\right)\left(x + \sqrt{5}\right) \geq 0$. *Case 1:* $x \geq \sqrt{5}$ and $x \geq -\sqrt{5}$, so $x \in \left[\sqrt{5}, \infty\right)$. *Case 2:* $x \leq \sqrt{5}$ and $x \leq -\sqrt{5}$, so $x \in \left(-\infty, -\sqrt{5}\right]$. Thus, the solution set is $\left(-\infty, -\sqrt{5}\right] \cup \left[\sqrt{5}, \infty\right)$.
Another method: $x^2 \geq 5 \quad \Leftrightarrow \quad |x| \geq \sqrt{5} \quad \Leftrightarrow \quad x \geq \sqrt{5}$ or $x \leq -\sqrt{5}$.



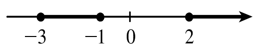
137. $x^3 - x^2 \leq 0 \quad \Leftrightarrow \quad x^2(x-1) \leq 0$. Since $x^2 \geq 0$ for all x, the inequality is satisfied when $x - 1 \leq 0 \quad \Leftrightarrow \quad x \leq 1$. Thus, the solution set is $(-\infty, 1]$.



138. $(x+1)(x-2)(x+3) = 0 \quad \Leftrightarrow \quad x = -1, 2$, or -3. Construct a chart:

Interval	$x+1$	$x-2$	$x+3$	$(x+1)(x-2)(x+3)$
$x < -3$	$-$	$-$	$-$	$-$
$-3 < x < -1$	$-$	$-$	$+$	$+$
$-1 < x < 2$	$+$	$-$	$+$	$-$
$x > 2$	$+$	$+$	$+$	$+$

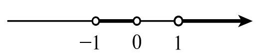
Thus, $(x+1)(x-2)(x+3) \geq 0$ on $[-3, -1]$ and $[2, \infty)$, and the solution set is $[-3, -1] \cup [2, \infty)$.



139. $x^3 > x \quad \Leftrightarrow \quad x^3 - x > 0 \quad \Leftrightarrow \quad x\left(x^2 - 1\right) > 0 \quad \Leftrightarrow \quad x(x-1)(x+1) > 0$. Construct a chart:

Interval	x	$x-1$	$x+1$	$x(x-1)(x+1)$
$x < -1$	$-$	$-$	$-$	$-$
$-1 < x < 0$	$-$	$-$	$+$	$+$
$0 < x < 1$	$+$	$-$	$+$	$-$
$x > 1$	$+$	$+$	$+$	$+$

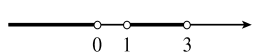
Since $x^3 > x$ when the last column is positive, the solution set is $(-1, 0) \cup (1, \infty)$.



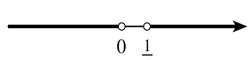
140. $x^3 + 3x < 4x^2 \quad \Leftrightarrow \quad x^3 - 4x^2 + 3x < 0 \quad \Leftrightarrow \quad x\left(x^2 - 4x + 3\right) < 0 \quad \Leftrightarrow \quad x(x-1)(x-3) < 0$.

Interval	x	$x-1$	$x-3$	$x(x-1)(x-3)$
$x < 0$	$-$	$-$	$-$	$-$
$0 < x < 1$	$+$	$-$	$-$	$+$
$1 < x < 3$	$+$	$+$	$-$	$-$
$x > 3$	$+$	$+$	$+$	$+$

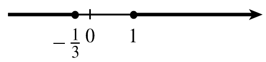
Thus, the solution set is $(-\infty, 0) \cup (1, 3)$.



141. $1/x < 4$. This is clearly true for $x < 0$. So suppose $x > 0$. then $1/x < 4 \quad \Leftrightarrow \quad 1 < 4x \quad \Leftrightarrow \quad \frac{1}{4} < x$. Thus, the solution set is $(-\infty, 0) \cup \left(\frac{1}{4}, \infty\right)$.



142. $-3 < 1/x \leq 1$. We solve the two inequalities separately and take the intersection of the solution sets. First, $-3 < 1/x$ is clearly true for $x > 0$. So suppose $x < 0$. Then $-3 < 1/x \quad \Leftrightarrow \quad -3x > 1 \quad \Leftrightarrow \quad x < -\frac{1}{3}$, so for this inequality, the solution set is $\left(-\infty, -\frac{1}{3}\right) \cup (0, \infty)$. Now $1/x \leq 1$ is clearly true if $x < 0$. So suppose $x > 0$. Then $1/x \leq 1 \quad \Leftrightarrow \quad 1 \leq x$, and the solution set here is $(-\infty, 0) \cup [1, \infty)$. Taking the intersection of the two solution sets gives the final solution set: $\left(-\infty, -\frac{1}{3}\right) \cup [1, \infty)$.



143. $C = \frac{5}{9}(F - 32) \quad \Rightarrow \quad F = \frac{9}{5}C + 32$. So $50 \leq F \leq 95 \quad \Rightarrow \quad 50 \leq \frac{9}{5}C + 32 \leq 95 \quad \Rightarrow \quad 18 \leq \frac{9}{5}C \leq 63 \quad \Rightarrow$ $10 \leq C \leq 35$. So the interval is $[10, 35]$.

144. Since $20 \leq C \leq 30$ and $C = \frac{5}{9}(F - 32)$, we have $20 \leq \frac{5}{9}(F - 32) \leq 30 \quad \Rightarrow \quad 36 \leq F - 32 \leq 54 \quad \Rightarrow$

$68 \leq F \leq 86$. So the interval is $[68, 86]$.

145. (a) Let T represent the temperature in degrees Celsius and h the height in km. $T = 20$ when $h = 0$ and T decreases by 10°C for every km (1°C for each 100-m rise). Thus, $T = 20 - 10h$ when $0 \leq h \leq 12$.

(b) From part (a), $T = 20 - 10h \quad \Rightarrow \quad 10h = 20 - T \quad \Rightarrow \quad h = 2 - T/10$. So $0 \leq h \leq 5 \quad \Rightarrow$ $0 \leq 2 - T/10 \leq 5 \quad \Rightarrow \quad -2 \leq -T/10 \leq 3 \quad \Rightarrow \quad -20 \leq -T \leq 30 \quad \Rightarrow \quad 20 \geq T \geq -30 \quad \Rightarrow$ $-30 \leq T \leq 20$. Thus, the range of temperatures (in $^\circ$C) to be expected is $[-30, 20]$.

146. The ball will be at least 32 ft above the ground if $h \geq 32 \quad \Leftrightarrow \quad 128 + 16t - 16t^2 \geq 32 \quad \Leftrightarrow$

$16t^2 - 16t - 96 \leq 0 \quad \Leftrightarrow \quad 16(t - 3)(t + 2) \leq 0$. $t = 3$ and $t = -2$ are endpoints of the interval we're looking for, and constructing a table gives $-2 \leq t \leq 3$. But $t \geq 0$, so the ball will be at least 32 ft above the ground in the time interval $[0, 3]$.

147. $|x + 3| = |2x + 1| \quad \Leftrightarrow \quad$ either $x + 3 = 2x + 1$ or $x + 3 = -(2x + 1)$. In the first case, $x = 2$, and in the second case, $x + 3 = -2x - 1 \quad \Leftrightarrow \quad 3x = -4 \quad \Leftrightarrow \quad x = -\frac{4}{3}$. So the solutions are $-\frac{4}{3}$ and 2.

148. $|3x + 5| = 1 \quad \Leftrightarrow \quad$ either $3x + 5 = 1$ or -1. In the first case, $3x = -4 \quad \Leftrightarrow \quad x = -\frac{4}{3}$, and in the second case, $3x = -6 \quad \Leftrightarrow \quad x = -2$. So the solutions are -2 and $-\frac{4}{3}$.

149. By Property 5 of absolute values, $|x| < 3 \quad \Leftrightarrow \quad -3 < x < 3$, so $x \in (-3, 3)$.

150. By Properties 4 and 6 of absolute values, $|x| \geq 3 \quad \Leftrightarrow \quad x \leq -3$ or $x \geq 3$, so $x \in (-\infty, -3] \cup [3, \infty)$.

151. $|x - 4| < 1 \quad \Leftrightarrow \quad -1 < x - 4 < 1 \quad \Leftrightarrow \quad 3 < x < 5$, so $x \in (3, 5)$.

152. $|x - 6| < 0.1 \quad \Leftrightarrow \quad -0.1 < x - 6 < 0.1 \quad \Leftrightarrow \quad 5.9 < x < 6.1$, so $x \in (5.9, 6.1)$.

153. $|x + 5| \geq 2 \quad \Leftrightarrow \quad x + 5 \geq 2$ or $x + 5 \leq -2 \quad \Leftrightarrow \quad x \geq -3$ or $x \leq -7$, so $x \in (-\infty, -7] \cup [-3, \infty)$.

154. $|x + 1| \geq 3 \quad \Leftrightarrow \quad x + 1 \geq 3$ or $x + 1 \leq -3 \quad \Leftrightarrow \quad x \geq 2$ or $x \leq -4$, so $x \in (-\infty, -4] \cup [2, \infty)$.

155. $|2x - 3| \leq 0.4 \quad \Leftrightarrow \quad -0.4 \leq 2x - 3 \leq 0.4 \quad \Leftrightarrow \quad 2.6 \leq 2x \leq 3.4 \quad \Leftrightarrow \quad 1.3 \leq x \leq 1.7$, so $x \in [1.3, 1.7]$.

156. $|5x - 2| < 6 \quad \Leftrightarrow \quad -6 < 5x - 2 < 6 \quad \Leftrightarrow \quad -4 < 5x < 8 \quad \Leftrightarrow \quad -\frac{4}{5} < x < \frac{8}{5}$, so $x \in \left(-\frac{4}{5}, \frac{8}{5}\right)$.

157. $a(bx - c) \geq bc \quad \Leftrightarrow \quad bx - c \geq \dfrac{bc}{a} \quad \Leftrightarrow \quad bx \geq \dfrac{bc}{a} + c = \dfrac{bc + ac}{a} \quad \Leftrightarrow \quad x \geq \dfrac{bc + ac}{ab}$

158. $ax + b < c \quad \Leftrightarrow \quad ax < c - b \quad \Leftrightarrow \quad x > \dfrac{c - b}{a}$ (since $a < 0$)

159. $|ab| = \sqrt{(ab)^2} = \sqrt{a^2 b^2} = \sqrt{a^2}\,\sqrt{b^2} = |a|\,|b|$

160. If $0 < a < b$, then $a \cdot a < a \cdot b$ and $a \cdot b < b \cdot b$ [using Rule 3 of Inequalities]. So $a^2 < ab < b^2$ and hence $a^2 < b^2$.